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Singular Mechanics and Landau Two-Fluid Model of Superfluidity

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We present a theoretical treatment of the Landau two-fluid model of superfluidity in liquid helium by means of the Dirac formalism. We introduce hydrodynamic considerations in a natural way by means of Lagrange multipliers. All constraints in phase space, in Dirac's sense, are second class and, as a consequence, the Dirac bracket differs strongly from the Poisson bracket. We calculate the Dirac bracket of the canonical variables, putting special interest on the density and the momentum density of the system. Our results generalize the results given by Dzyaloshinskii and Volovik and correct other published results.

1. INTRODUCTION

This paper presents a study of the Landau two-fluid model (Putterman, 1974) of superfluidity based on the Dirac formalism for singular systems (Dirac, 1950, 1951, 1958, 1964; Sudarshan and Mukunda, 1974).

As is well known, the Lagrangian formulation of the equations of motion of the two-fluid model is due to Khalatnikov (1952) and the equivalent Hamiltonian formulation was established by Pokrovsky and Khalatnikov (1976). It is important to stress the fact that these authors do not take into account the singularity of the Lagrangian density, i.e., the existence of constraints in phase space. Due to this fact, we have to follow Dirac and apply his formalism to find the time evolution of the physical variables.

Rodríguez-Gómez (1980) studied superfluid helium by means of a Dirac treatment for singular systems, but he obtains results different from the ones we present here. Additionally, we have calculated the Dirac bracket

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of the relevant physical variables, which was not done in Rodríguez-Gómez (1980).

In Section 2 we review the main basic concepts and consequences of the formalism to be used. Section 3 is dedicated to the Lagrangian density proposed by Khalatnikov to study superfluid helium as a two-fluid system.

We apply the Dirac formalism to our Lagrangian density in Section 4. We give the conclusions of this work in Section 5.

2. A SURVEY OF DIRAC MECHANICS

We write down the main results of the Dirac formalism which are connected with the present work.

Let $q=(q_1,\ldots,q_N)$ and $p=(p_1,\ldots,p_N)$ be the generalized coordinates and momenta of a system of N particles. A Lagrangian is defined to be singular if and only if

$$\det|\partial^2 L/\partial q_i\, \partial q_j| = 0 \tag{2.1}$$

where $i, j = 1, 2, \ldots, N$, and L is the Lagrangian of the system.

The primary constraints[3] $f_\alpha^1(q,p) \cong 0 (\alpha = 1, 2, \ldots, M)$ are those which arise from the definition of the canonical momentum

$$p_n \equiv \partial L/\partial q_n \tag{2.2}$$

and the secondary constraints $f_\beta^\gamma(q,p) \cong 0$, with $\gamma \geq 2$, are those which result from the self-consistent equations

$$f_\beta^\gamma = \partial f_\beta^\gamma/\partial t + q_n\, \partial f_\beta^\gamma/\partial q_n + p_n\, \partial f_\beta^\gamma/\partial p_n \cong 0 \tag{2.3}$$

where we have adopted the Einstein convention for repeated indices. The equations of motion for q_n and p_n are given by

$$q_n = \partial H/\partial p_n + U_\alpha\, \partial f_\alpha^1/\partial p_n \tag{2.4}$$

$$-p_n = \partial H/\partial q_n + U_\alpha\, \partial f_\alpha^1/\partial q_n \tag{2.5}$$

The summation in α is for the primary constraints. The U_α are noncanonical variables and maybe obtained from self-consistency (or noncontradiction) equations.

Along the same lines, a variable $g(p,q)$ is first class if

$$\{g, f_\alpha^\beta\}_- \cong 0, \qquad \forall \alpha \tag{2.6}$$

If g does not satisfy equation (2.6), it is a second-class variable. Dirac also proves the following theorem.

[3] $f_\alpha^{(1)}$ is given by $p_\alpha - \partial L/\partial q_\alpha \cong 0$.

Theorem. A quantity $h(q, p)$ is a physical variable if and only if

$$\{h, f_\alpha\}_- \cong 0 \tag{2.7}$$

f_α refers to all constraints. If h does not satisfy equation (2.7), we call it a nonphysical variable.

The Dirac formalism leads to the construction of the so-called Dirac bracket given by the following expression:

$$\{g, h\}^*_- = \{g, h\}_- - \{g, f_\alpha\}_- C^{\alpha\beta}\{f_\beta, h\}_- \tag{2.8}$$

where we are summing over one irreducible decomposition of second-class constraints, $C^{\alpha\beta}$ is the inverse matrix of the Poisson brackets $\{f_\alpha, f_\beta\}$, and $\{\cdot, \cdot\}_-$ stands for the antisymmetric Poisson bracket.

If we follow the work of Sundarshan and Mukunda, we can easily generalize the present formalism to the case of field theory.

3. LANDAU TWO-FLUID MODEL

Landau proposed a macroscopic model for liquid helium consisting of a mixture of two fluids. According to him, in liquid helium, two movements can exist simultaneously, each having its own "effective mass." One of these motions corresponds to a normal liquid and the other is associated with the superfluidity properties of helium below a critical temperature. Both motions take place without a momentum transfer from one liquid to the other. In a certain sense, we may talk of "superfluid" and "normal" masses of liquid helium. However, we should keep in mind that real division of the liquid into two parts is impossible.

During a rotation of a vessel containing helium, the superfluid part remains stationary. This means that $\nabla \times V_s = 0$, where V_s is the superfluid velocity. The part which is responsible for the viscosity, the normal fluid, is characterized by the normal velocity, V_n.

The Lagrangian density proposed by Khalatnikov is

$$\begin{aligned}\mathscr{L} = -\rho V_s^2/2 + J\cdot V_s - \varepsilon(\rho, S, V_n - V_s) + \alpha(\rho + \nabla\cdot J)\\ + \beta[S + \nabla\cdot(SV_n)] + \nu[F + \nabla\cdot(FV_n)]\end{aligned} \tag{3.1}$$

where

$$J = \rho_n V_n + \rho_s V_s = \rho_n(V_n - V_s) + \rho V_s \tag{3.2}$$

and ρ_n and ρ_s are the densities of the two parts of the liquid helium; S is the entropy density; ε is the energy density; and J is the total momentum density of the liquid.

We immediately observe in equation (3.1) that the normal part is responsible for the energy transfer. We also conclude that the parameters α, β, and ν are Lagrange multipliers. As a direct consequence, the equations of motion which result from the variations of α, β, and ν are the continuity equation, the entropy production. S, β, F, and ν are Clesch variables needed to describe the momentum of the liquid (Seliger and Whitham, 1969).

It is important to point out that the additions of a total derivative or gauge transformation, according to the notation of Levy-Leblond (1969), has no consequences in a quantized theory coming from a singular Lagrangian density, as proved by Tello-Llanos (1984) and much more recently by Rodríguez-Núñez (1990). The last work refers to the case of fermionic variables at the quantum level or Grassman variables (Negele and Orland, 1988) at the classical one.

Due to their association in the Lagrangian density, the canonical variables are

$$(\rho, \alpha), \quad (S, \beta), \quad \text{and} \quad (F, \nu) \tag{3.3}$$

Variations with respect to V_n yield

$$J = \rho \nabla \alpha + S \nabla \beta + F \nabla \nu = \rho V_s + p \tag{3.4}$$

where

$$V_s \equiv \nabla \alpha; \qquad p \equiv S \nabla \beta + F \nabla \nu = \rho_n (V_n - V_s) \tag{3.5}$$

Using the last two equations, we can rewrite the Lagrangian density as

$$\mathscr{L} = \alpha \rho + \beta S + \nu F - \rho V_s^2/2 - \varepsilon_N(\rho, S, p) - p \cdot V_s \tag{3.6}$$

where

$$\varepsilon_N \equiv \varepsilon + p \cdot (V_n - V_s) \tag{3.7}$$

Equation (3.6) was derived using the fact that both the normal and superfluid flows have zero velocity perpendicular to the container (Landau and Lifshitz, 1987).

4. APPLICATION OF THE DIRAC FORMALISM

From equation (3.1), we easily obtain the primary constraints. They are

$$f_1^1 = P_\rho - \alpha \cong 0 \tag{4.1}$$

$$f_2^1 = P_s - \beta \cong 0 \tag{4.2}$$

$$f_3^1 = P_F - \nu \cong 0 \tag{4.3}$$

$$f_4^1 = P_\alpha \cong 0 \tag{4.4}$$

$$f_5^1 = P_\beta \cong 0 \tag{4.5}$$

$$f_6^1 = P_\nu \cong 0 \tag{4.6}$$

The total Hamiltonian density $\mathscr{H}_T$ is given by

$$\mathscr{H}_T = \rho V_s^2/2 + p\cdot V_s + \varepsilon_N(\rho, S, p) + U_1(P_\rho - \alpha) + U_2(P_s - \beta) + U_3(P_F - \nu) + U_4 P_\alpha + Y_5 P_\beta + U_\sigma P_\nu \tag{4.7}$$

The equations of motion are

$$\rho = \delta\mathscr{H}_T/\delta P_\rho = U_1 \tag{4.8}$$

$$S = \delta\mathscr{H}_T/\delta P_s = U_2 \tag{4.9}$$

$$F = \delta\mathscr{H}_T/\delta P_F = U_3 \tag{4.10}$$

$$\alpha = P_\rho = -\delta\mathscr{H}_T/\delta\rho = -(\mu + V_s^2/2) \tag{4.11}$$

$$\beta = P_S = -\delta\mathscr{H}_T/\delta S = -(T + V_n\cdot\nabla\beta) \tag{4.12}$$

$$\nu = P_F = -\delta\mathscr{H}_T/\delta F = -V_n\cdot\nabla_\nu \tag{4.13}$$

The evolution of α, β, ν, P_α, P_β, and P_ν is given by

$$\alpha = U_4; \qquad \rho + \nabla\cdot J = 0 \tag{4.14}$$

$$\beta = U_5; \qquad S + \nabla\cdot(SV_n) = 0 \tag{4.15}$$

$$\nu = U_6; \qquad F + \nabla\cdot(FV_n) = 0 \tag{4.16}$$

From our previous equations, we obtain the constraint equations imposed at the Lagrangian level plus the time evolution for α, β, and ν. Equation (4.11) implies that the superfluid is a potential flow (Landau and Lifshitz, 1987).

To determine the secondary constraints, we must use the noncontradiction equation (Rodríguez-Núñez, 1977)

$$\{f_k^{(1)}(x), \mathscr{H}_T\} + \sum_i \int d^3x'\, U_i(x')\{f_k^{(1)}(x), f_i^{(1)}(x')\}_- \cong 0 \tag{4.17}$$

Before evaluating equation (4.17), we write down the Poisson brackets of the primary constraints. They are

$$\{f_1^{(1)}, f_4^{(1)}\}_- = \{f_2^{(1)}, f_5^{(1)}\}_- = \{f_3^{(1)}, f_6^{(1)}\}_- = \delta^{(3)}(x - x') \tag{4.18}$$

and zero for the other terms.

When we apply equation (4.17) to our primary constraints, we do not obtain new constraint equations. So, secondary constraints are missing in this particular example. Also, our constraints are of second class according to the Dirac notation. The last result implies that

$$\{F, G\}_- \neq \{F, G\}_-^* \tag{4.19}$$

i.e., the Dirac bracket is different from the Poisson bracket. This is a big difference from the results given in Rodríguez-Gómez (1980).

It is an easy matter to show that

$$\{\rho(x), \alpha(x')\}^*_- = \delta^{(3)}(x-x') \tag{4.20}$$

$$\{\rho(x), \rho(x')V_{sj}(x')\}^*_- = -\rho(x')\partial_{x_j}\delta^{(3)}(x-x') \tag{4.21}$$

$$\begin{aligned}\{p_i(x), p_j(x')\}^*_- = {} & (S(x)\partial_{x_{j'}}\beta(x') + F(x)\partial_{x_{j'}}\nu(x'))\partial_{x_{i'}}\delta(x-x') \\ & -(S(x')\partial_{x_i}\beta(x) + F(x')\partial_{x_i}\nu(x)) \\ & \times \partial_{x_j}\delta^{(3)}(x-x')\end{aligned} \tag{4.22}$$

$$\{S(x), \beta(x')\}^*_- = \delta^{(3)}(x-x') \tag{4.23}$$

$$\{F(x), \nu(x')\}^*_- = \delta^{(3)}(x-x') \tag{4.24}$$

In passing, let us recall that the Theorem given by equation (2.7) applies to our system, since our constraints are second class. So, *all* variables are physical.

5. CONCLUSIONS

For the first time, we have been able to show a consistent way to handle the superfluid constraints in phase space. We have obtained the Dirac brackets for all canonical variables and they satisfy the appropriate commutation rules of a field theory, as they should.

Also, from a classical point of view, we have calculated the Dirac brackets $\{\rho, J\}^*_-$ and $\{p_i, p_j\}^*_-$. The expressions we obtain are the correct results for a classical field theory. It is worthwhile to mention that Dzyaloshinskii and Volovik (1980) postulated the classical Poisson brackets of the same field quantities. However, their results have the shortcoming of being derived from quantum mechanics in the classical limit. We should recall Dirac when he says that we have to be very careful when going either from classical to quantum theory or the other way around.

Another conclusion we have reached is that all our field variables are physical, i.e., they are needed to describe the helium superfluid properties. Of course, $p = S\nabla\beta + F\nabla\nu$ is a physical variable, since it is a combination of physical ones. As a consequence, the theorem of Pokrovsky and Khalatnikov (1976) retains its full validity in our formulation.

In a simple fashion we can get the conservation equations (Rodríguez-Gómez, 1980). Besides these features, we have allowed vorticity contributions with $\nabla \times p \neq 0$. These aspects are covered from a group theory point of view by Khalatnikov and Lebedev (1978).

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